EXTREMES OF ORDER STATISTICS OF SELF-SIMILAR PROCESSES

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Abstract: Let $\{X_i(t), t \geq 0\}, 1 \leq i \leq n$ be independent copies of a self-similar process $\{X(t), t \geq 0\}$. For given positive constants u, T, define the set of rth conjunctions $C_{r,T,u} := \{t \in [0,T], X_{r:n}(t) \geq u\}$ with $X_{r:n}(t)$ the rth largest order statistics of $X_1(t), \ldots, X_n(t), t \geq 0$. In numerous applications such as brain mapping and digital communication systems, of interest is the approximation of the probability that the set of conjunctions $C_{r,T,u}$ is not empty. In this paper, we obtain, by imposing the Albin's Conditions on X, an exact asymptotic expansion of this probability as u tends to infinity as well as the asymptotic tail distributions of the mean sojourn time of $X_{r:n}$ over an increasing interval. Further, we explain our results by some examples concerning bi-fractional Brownian motion, sub-fractional Brownian motion and the generalized skew Gaussian process.
Key Words: Self-similar processes; order-statistic processes; conjunction; extremes; mean sojourn time; generalized skew-Gaussian process; bi-fractional Brownian motion; sub-fractional Brownian motion.

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1. Introduction

Let $\{X(t), t \geq 0\}$ be a self-similar process with index $\kappa > 0$ and P-continuous sample paths, i.e., the finite-dimensional distributions (f.d.d.) of $X(\lambda t)$ coincide with those of $\lambda^\kappa X(t)$ for all $\lambda > 0$. Denote by $X_1, \ldots, X_n, n \in \mathbb{N}$ independent copies of X. The main object of interest in this contribution is the rth order statistics process $X_{r:n}$ defined from $X_1, \ldots, X_n$ in the usual way, i.e., for any $t \geq 0$

$$X_{n:n}(t) \leq \cdots \leq X_{1:n}(t). \tag{1}$$

For a given positive threshold u and a fixed positive constant T define the set of rth conjunction $C_{r,T,u}$ by

$$C_{r,T,u} := \{t \in [0,T] : X_{r:n}(t) \geq u\}$$

and set

$$p_{r,T}(u) := \mathbb{P}\{C_{r,T,u} \neq \phi\} = \mathbb{P}\left\{ \sup_{t \in [0,T]} X_{r:n}(t) \geq u \right\}. \tag{2}$$

Of interest is the calculation of $p_{r,T}(u)$ in various applications, for instance, for the analysis of functional magnetic resonance imaging (fMRI) data and the surface roughness during all machinery processes. For smooth Gaussian random fields approximations of $p_{n,T}(u)$ are discussed for instance in [8, 16, 24]; results for non-Gaussian random fields can be found for instance in [10]. The recent contributions [17, 19] derived asymptotic expansions of $p_{r,T}(u)$

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considering a stationary (Gaussian) process X. It is well-known that the stationary random field cannot be used to model phenomena and data sets that exhibit certain non-stationary characteristics such as long-range dependence (LRD). Such situations arise naturally in limit theorems of random walks and other stochastic processes, and they have been applied to model various phenomena in a wide range of scientific areas including telecommunications, internet traffic, image processing and mathematical finance. These processes are always related to self-similar processes such that the scaling of time is equivalent to an appropriate scaling of space. We refer to the monographs [23, 20, 11, 12] for complete expositions on theoretical and practical aspects of self-similar processes.

Motivated by [5, 19] and the tractability of self-similar processes, in this paper, we shall investigate the asymptotic behaviour of $p_{r,T}(u)$ as $u \to \infty$ and T fixed. The main methodology employed here is from Patrik Albin [5]. As mentioned therein, there are no systematic approaches to non-stationary extremes comparable with stationary theories. Albin's theory of extremes for self-similar non-stationary processes, concerning checking a couple of conditions can be developed that performs (at least) as well as stationary counterparts.

The main contributions of this paper are three-folded. The first is the extensional results of the mean sojourn time of X into those of rth order statistics process $X_{r:n}$ (see below Propositions 2.1, 2.2). The second is that the asymptotic results of the survival probability of $\sup_{t\in[0,1]} X_{r:n}(t)$ over an increasing interval (see Theorems 3.1–3.4) holds under some common conditions imposed on X. Finally, several examples, such as Gaussian processes (see, e.g. bi-fractional Brownian motion, sub-fractional Brownian motion) and non-Gaussian processes (see, e.g., the generalized self-similar skew-Gaussian process), are considered to utilize our results (see, Theorems 4.1, 4.2).

This paper is organized as follows: In Section 2 we state our main conditions and some preliminary results. In Section 3, we present the main results following by some examples and an application concerning a generalized self-similar skew-Gaussian processes. In Section 5 we present all the proofs of our results.

2. Preliminaries and Technical Conditions for Self-similar Processes

In this section, we state first the so-called Conditions A, B, C and C$'$ imposed on the self-similar process X, which are extremely useful to formulate the extreme properties of X; see, e.g., [5], and then we establish two asymptotic preliminaries for the mean sojourn time of $X_{r:n}$ over an increasing interval, which are new and useful to understand our main results in Section 3.

Throughout this paper, we consider $\{X(t), t \ge 0\}$ to be the self-similar process X with index $\kappa > 0$ and P-continuous sample paths, and the rth order statistics process $X_{r:n}$ is given as in (1) generated by X, and assume that the distribution function (df) of $X(1)$, denoted by G, has an infinite right endpoint and continuous at infinity. Let J be an interval such that $J \subseteq (-1, \infty)$ with $0 \in J$, and $q = q(u)$ a non-increasing, positive function such that $Q = 1/\lim_{u\to\infty} q(u)$ exists and $\tilde{a} = 1/(2\sup_{u<\infty} q(u)) > 0$. The function q and the notation $Q, \tilde{a}$ are depicted in all conditions and theorems.

Condition A: (Gumbel MDA and Conditional limit distribution) Assume that G belongs to the Gumbel max-domain of attraction (MDA), denoted by $G \in D(\Lambda)$, that is, there exists some positive measurable function $w = w(u)$ such that

$$\lim_{u\to\infty} \frac{1-G(u+x/w)}{1-G(u)} = e^{-x}, \quad x \in \mathbb{R}. \tag{3}$$

Further, there exists an $(\mathbb{R}\cup\{-\infty,\infty\})$-valued process $\{\xi(t), t\geq 0\}$ such that

$$\lim_{u\to\infty} \mathbb{P}\left\{\bigcap_{i=1}^{m} \{w\big(X(1-qt_i)-u\big) > x_i\}\Big| X(1) > u\right\} = \mathbb{P}\left\{\bigcap_{i=1}^{m}\{\xi(t_i) > x_i\}\right\} \tag{4}$$

holds for $m \in \mathbb{Z}^+, t_i \in [0, Q)$ and continuity points $x_i \in J$ for the functions $\mathbb{P}\{\xi(t_i) > \cdot\}, i = 1,\ldots,m$, respectively. The assumption that $G \in D(\Lambda)$ implies that $\lim_{u\to\infty} 1/(uw(u)) = 0$ and that w is self-neglecting, i.e., the limit holds locally uniformly for $x \in \mathbb{R}$ that (see, e.g., [18])

$$\lim_{u\to\infty} \frac{w(u+x/w(u))}{w(u)} = 1. \tag{5}$$

Condition B: (Short-lasting-exceedance) We have

$$\lim_{d\to\infty} \limsup_{u\to\infty} \int_{d\wedge(1/q)}^{1/q} \mathbb{P}\{X(1-qt) > u | X(1) > u\}\, dt = 0. \tag{6}$$

Condition B is void if $Q < \infty$, and if Condition A holds then Condition B can be interpreted as $\mathbb{E}\left(\int_0^Q \mathbb{I}_{(u,\infty)}(\xi(t))\, dt\right)$ where $\mathbb{I}_S(\cdot)$ is the indicator function of a set S. More generally, Proposition 2 in [5] showed that it holds when $\beta_3 > 0$ (see (10) for a precise definition). Condition B is developed to be used for establishing the tail asymptotic of stationary processes under consideration; see, e.g., [13, 2, 3, 7].

Moreover, Propositions 2.1, 2.2 below show that Conditions A, B imposed on X hold also for the order statistics process $X_{r:n}$, which will be utilized to establish the lower bound of extremes; see Theorem 3.1.

To state the two tightness Conditions C and C$'$ used for establishing the upper bound of extremes, we denote $t_a^u(0) = 1$ and

$$t_a^u(k+1) = t_a^u(k)\big(1 - aq(t_a^u(k)^{-\kappa}u)\big)$$

for $k \leq K \equiv K(a,u) = \sup\{k \in \mathbb{N} : t_a^u(k)^{-\kappa}u < \infty\}$ where $u \in \mathbb{R}, a \in (0, \tilde{a}]$ are given.

Condition C: For some choice of $\sigma > 0$ and $a \in (0, \tilde{a}]$, we have

$$\boldsymbol{v}(a,\sigma) \equiv \limsup_{u\to\infty} \frac{\mathbb{P}\left\{\sup_{t\in[0,1]} X(t) > u + \sigma/w, \max_{0\leq k\leq K} X(t_a^u(k)) \leq u\right\}}{\mathbb{E}\left(L(u)/q\right) + \mathbb{P}\{X(1) > u\}} < \infty,$$

with $L(u) = \int_0^1 \mathbb{I}_{(u,\infty)}(X(t))\, dt$, the mean sojourn time of X over the threshold u on $[0,1]$.

Condition C$'$: Condition C holds with $\lim_{a\downarrow 0} \boldsymbol{v}(a,\sigma) = 0$ for each $\sigma > 0$.

Conditions C and C$'$ are often verified via Propositions 3–5 in [5], which are basically a few estimates related to the tail behavior of the one- and two- dimensional distributions of the involved process. For convenience, we state Proposition 3 (ii) below, denoted by Condition C*, which is a sufficient condition of Condition C$'$.

Condition C*: Suppose that there exist positive constants λ_0, ρ, b, D and $d > 1$ such that

$$\mathbb{P}\left\{X(1-qt) > u + \frac{\lambda + v}{w}, X(1) \le u + \frac{v}{w}\right\} \le Dt^d \lambda^{-b} \mathbb{P}\{X(1) > u\}$$

holds for all u large and all $0 < t^\rho < \lambda \le \lambda_0, v \ge 0$.

In our setting, most results require that the function $p(u) = u^{-1/\kappa} q(u)$ satisfies that the limit $\hat{p}(x)$ given by

$$\hat{p}(x) = \lim_{u\to\infty} \frac{p(u + x/w)}{p(u)} \tag{7}$$

exists and is continuous for $x > 0$.

In addition, we need another two requirements that there is a $\rho \ge 0$ such that

$$\beta_1 = \liminf_{v\to\infty} \inf_{u\in[v,\infty]} \frac{u^\rho p(u)}{v^\rho p(v)} > 0, \tag{8}$$

and that

$$\beta_2 = \limsup_{v\to\infty} \sup_{u\in[v,\infty]} \frac{vw(v)}{uw(u)} < \infty. \tag{9}$$

Generally, (7), (8) hold when for instance q is non-increasing and regular varying at infinity. While (9) is natural since $\lim_{u\to\infty} 1/(uw(u)) = 0$. Finally, the behaviors of extremes will depend on whether the limits

$$\beta_3 = \liminf_{u\to\infty} uq(u)w(u) \quad \text{and} \quad \beta_4 = \limsup_{u\to\infty} uq(u)w(u) \tag{10}$$

are finite or infinite; see, e.g., Theorem 3.3 and Theorem 3.4 in Section 3.

Next, we establish the following generalizations of Proposition 1 and Theorem 1 in [5] concerning the asymptotic properties of $L_r(u)$, the mean sojourn time of rth order statistics process $X_{r:n}$ over threshold u, that is,

$$L_r(u) \equiv L_r(1; u), \quad L_r(s; u) = \int_0^s \mathbb{I}_{(u,\infty)}(X_{r:n}(t))\, dt, \quad s \in [0, 1]. \tag{11}$$

Proposition 2.1. *Suppose that $G \in D(\Lambda)$ with the auxiliary function w. Then, the df of $X_{r:n}(1)$ belongs to the Gumbel MDA with the auxiliary function $w_r(u) = rw(u)$, and*

$$\mathbb{E}(L_r(u)) = \frac{1}{r} \frac{1}{\kappa uw} \mathbb{P}\{X_{r:n}(1) > u\}(1 + o(1)), \quad u \to \infty. \tag{12}$$

More generally, we can obtain below the bounds of the asymptotic distribution of $L_r(u)$ if Conditions A, B and (7), (8) are satisfied by the generated process X. Let therefore $\{\xi_{r:r}(t), t \ge 0\}$ be the minimum process of r independent copies $\xi_1, \ldots, \xi_r$ of the random process ξ, and with $\hat{p}(s)$ given by (7)

$$\Theta_r(x) \equiv r \int_0^\infty \mathbb{P}\left\{\int_0^Q \mathbb{I}_{(u,\infty)}(\xi_{r:r}(t))\, dt > \frac{x}{\hat{p}(s)}\right\} e^{-rs}\, ds, \quad x \ge 0. \tag{13}$$

Proposition 2.2. *Suppose that Condition A and (7) hold for the process X. Then, for each $x \ge 0$ we have*

$$\liminf_{u\to\infty} \int_x^\infty \frac{\mathbb{P}\{L_r(u)/q > y\}}{\mathbb{E}(L_r(u)/q)}\, dy \ge \Theta_r(x).$$

If additionally Condition B and (8) hold, then for each $x \ge 0$ we have

$$\limsup_{u\to\infty} \int_x^\infty \frac{\mathbb{P}\{L_r(u)/q > y\}}{\mathbb{E}(L_r(u)/q)}\, dy \le \Theta_r(x-).$$

Remark 2.3. *Note that, in view of Propositions 2.1, 2.2, we have*

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t) > u\right\} &\ge \max\left(\mathbb{P}\left\{X_{r:n}(1) > u\right\}, \frac{1}{x}\int_0^x \mathbb{P}\left\{L_r(u)/q > y\right\} dy\right)\\ &\ge \max\left(\mathbb{P}\left\{X_{r:n}(1) > u\right\}, \frac{1-\Theta_r(x-)}{x}\mathbb{E}\left(L_r(u)/q\right)\right), \quad x>0.\end{aligned}$$

3. Main Results

In this section, we shall establish our main results which extend those for the self-similar process X. The first theorem is concerned to the lower bound for $\mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t) > u\right\}$ without knowing the size of $uq(u)w(u)$. While the remaining three theorems require some knowledge of it.

Theorem 3.1. *Suppose that Conditions A and B hold, and that* (7) *holds. Then we have*

$$\liminf_{u\to\infty} \frac{1}{\mathbb{E}\left(L_r(u)/q\right) + \mathbb{P}\left\{X_{r:n}(1) > u\right\}} \mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t) > u\right\} > 0.$$

Next, we establish the upper bound of extremes, which requires the tightness Condition C and the knowledge that $\beta_3 > 0$.

Theorem 3.2. *Suppose that Condition C and $\beta_3 > 0$ are satisfied for the self-similar process X. If further* (8) *holds, then we have*

$$\limsup_{u\to\infty} \frac{1}{\mathbb{E}\left(L_r(u)/q\right) + \mathbb{P}\left\{X_{r:n}(1) > u\right\}} \mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t) > u\right\} < \infty.$$

From Proposition 2.1 and Theorem 3.1, Theorem 3.2 above, $\mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t) > u\right\}$ could behaves like $\mathbb{P}\left\{X_{r:n}(1) > u\right\}$ or $\mathbb{E}\left(L_r(u)/q\right)$, which finally depends on $uq(u)w(u)$. Next, we shall establish the two sharp extremes depends on that β_4 is infinite and finite, respectively.

Theorem 3.3. *Suppose that Condition C′ and $\beta_3 > 0$ are satisfied for the self-similar process X. Assume further that* (8) *holds, then we have*

$$\beta_4 = \infty \quad \Longrightarrow \quad \liminf_{u\to\infty} \frac{1}{\mathbb{P}\left\{X_{r:n}(1) > u\right\}} \mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t) > u\right\} = 1$$

$$\beta_3 = \infty \quad \Longrightarrow \quad \lim_{u\to\infty} \frac{1}{\mathbb{P}\left\{X_{r:n}(1) > u\right\}} \mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t) > u\right\} = 1.$$

Theorem 3.4. *Suppose that Conditions A, C′ and $\beta_3 > 0$ are satisfied for the self-similar process X. Assume additionally that* (7)–(9) *and $\beta_4 < \infty$ hold. Then we have*

$$\lim_{u\to\infty} \frac{1}{\mathbb{E}\left(L_r(u)/q\right)} \mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t) > u\right\} = \lim_{x\downarrow 0} \frac{1-\Theta_r(x)}{x} := -\Theta_r'(0)$$

exists with $\Theta_r'(0) \in (0,\infty)$.

Corollary 3.5. *Theorem 3.4 still hold if Condition C' and $\beta_3 > 0$ are replaced by Conditions B and C^*.*

Remark 3.6. *a) Since $X_{r:n}$ is self-similar, Theorems 3.1–3.4 can be easily rewritten for $p_{r,T}(u)$.*

b) Most of self-similar processes satisfy the conditions required in the above theorems; see for instance the fractional Brownian motion, totally skewed linear fractional α-stable motion and Rosenblatt process in [5].

c) We can drop the assumption on $G \in D(\Lambda)$ with infinite endpoint to the general cases that G belonging to the three max-domain of attractions; see, e.g., [3, 5].

4. Examples and Applications

Examples of Gaussian process X Several important examples of Gaussian processes are self-similar processes. We present below two interesting Gaussian processes (see, e.g., [14, 15]):

Bi-fractional Brownian motion: Consider $\{B_{h,k}, t \ge 0\}$ with $h \in (0,1), k \in (0,1]$ to be a bi-fBm, i.e., a centered self-similar Gaussian process with covariance function given by

$$\mathbb{E}\left(B_{h,k}(t)B_{h,k}(s)\right) = \frac{1}{2^k}\left(\left(t^{2h} + s^{2h}\right)^k - |t-s|^{2hk}\right), \quad t, s \ge 0.$$

In particular, the bi-fBm $B_{h,1}$ is the fBm with Hurst index h. It follows by routine calculations that the Lamperti's transformation $\tilde{B}_{h,k}(t) \equiv e^{-\kappa t} B_{h,k}(e^t)$ with $\kappa = hk$ is a centered stationary Gaussian process with covariance function satisfying that

$$\mathbb{E}\left(\tilde{B}_{h,k}(0)\tilde{B}_{h,k}(t)\right) = 1 - 2^{-k} t^{2hk} + o(|t| + |t|^{2hk}), \quad t \to 0.$$

Sub-fractional Brownian motion: The sub-fBm $\{S_h(t), t \ge 0\}$ with $h \in (0,1)$ is a centered self-similar Gaussian process with covariance given by

$$\mathbb{E}\left(S_h(t)S_h(s)\right) = t^{2h} + s^{2h} - \frac{1}{2}\left((t+s)^{2h} + |t-s|^{2h}\right), \quad t, s \ge 0.$$

It follows by routine calculations that the Lamperti's transformation $\tilde{S}_h(t) \equiv e^{-\kappa t} S_h(e^t)$ with $\kappa = h$ is a centered stationary Gaussian process with covariance function satisfying that

$$\mathbb{E}\left(\tilde{S}_h(0)\tilde{S}_h(t)\right) = (2 - 2^{2h-1})\left(1 - \frac{1}{2(2-2^{2h-1})} t^{2h} + o(|t| + |t|^{2h})\right), \quad t \to 0.$$

Note that the correlation functions of the above centered stationary Gaussian processes have regular varying tails at zero. Using the well-known results for stationary Gaussian processes (see, e.g., [21, 6]) and the Lamperti's propositions in [5], it is easy to show that the above two self-similar Gaussian processes satisfy the conditions of Theorems 3.1–3.4; see also Theorems 4.1, 4.2 below.

The generalized self-similar skew-Gaussian processes: Recently, the skew-Gaussian processes have received a lot of attentions from both theoretical and applicable fields; see, e.g., [1, 9, 19]. Next, we will consider this non-Gaussian self-similar process and establish the tail asymptotic results by using our theorems in Section 3. The main methodology used here is to first analyze the Lamperti's associated stationary process , and then transfer the results to itself via Propositions 6–9 in [5].

Let $\{\chi(t), t \geq 0\}$ be a centered self-similar Gaussian process with index $\kappa > 0$ and covariance satisfies

$$\mathbb{E}\left(\chi(1)\chi(1+t)\right) = 1 + \kappa t - D\left|t\right|^{\alpha} + o(|t| + |t|^{\alpha}), \quad t \to 0 \tag{14}$$

for some constant $\alpha \in (0,2]$ and $D > 0$. Note that the classes of random processes satisfying (14) are very big, for instance the bi-fBm and the sub-fBm above. In particular, (14) holds with $D = 1/2, \kappa = \alpha/2$ if $\chi(t) = Z(t)$, a standard fBm with Hurst index $\alpha/2 \in (0,1]$ and $\mathbf{Cov}(Z(s), Z(t)) = 2^{-1}\left(s^{\alpha} + t^{\alpha} - |s-t|^{\alpha}\right), \quad s, t \geq 0$.

Denote by $\chi_i, i \leq m+1, m \in \mathbb{N}$ independent copies of χ, then our generalized skew Gaussian process $\zeta_{m,\delta}, \delta \in [0,1]$ is defined as (set $|\boldsymbol{\chi}(t)| := \delta\big(\sum_{i=1}^m \chi_i^2(t)\big)^{1/2}$)

$$\zeta_{m,\delta}(t) \equiv \delta\left|\boldsymbol{\chi}(t)\right| + \sqrt{1-\delta^2}\chi_{m+1}(t), \quad t \geq 0. \tag{15}$$

Therefore, $\zeta_{m,\delta}$ is associated to the stationary process $\tilde{\zeta}_{m,\delta}$ via the Lamperti's transformation such that $\tilde{\zeta}_{m,\delta}(t) := \delta\left|\tilde{\boldsymbol{\chi}}(t)\right| + \sqrt{1-\delta^2}\tilde{\chi}_{m+1}(t)$, which has independent standardized Gaussian components $\tilde{\chi}_i(t), i \leq m+1$ drawn from $\tilde{\chi}(t) = e^{-\kappa t}\chi(e^t)$ satisfying

$$\mathbb{E}\left(\tilde{\chi}(0)\tilde{\chi}(t)\right) = 1 - D\left|t\right|^{\alpha} + o(|t| + |t|^{\alpha}), \quad t \to 0. \tag{16}$$

As for the χ-process $\zeta_{m,1}$ studied by [5], we need to impose further a bound condition on the covariance function (14): There exists some $h > 0$ such that

$$\sup_{t \in [\varepsilon, h]} e^{-\kappa t}\mathbb{E}\left(\chi(1)\chi(e^t)\right) < 1 \quad \text{for } \varepsilon \in (0, h]. \tag{17}$$

Note that the stationary Gaussian process satisfying (16) is studied first by [21], and then is extended to the $\tilde{\zeta}_{m,1}$ was by [22]. Our results for $\zeta_{m,\delta}$ and the results on its mean sojourn time below are new for $m \geq 1$ and $\delta \in (0,1)$. For notational simplicity, let E be a unit exponential random variable (rv) which is independent of the standard fBm Z with Hurst index $\alpha/2 \in (0,1]$. And $Z_i, i \leq r$ and $E_i, i \leq r$ are r independent copies of Z and E, respectively.

Theorem 4.1. *Let $\{\zeta_{m,\delta}(t), t \geq 0\}$ be defined by (1) satisfying (14) and (17) with $\alpha \in (0,1]$. Let $X_{r:n}$ be generated by $\zeta_{m,\delta}, \delta \in [0,1]$. Then Propositions 2.1, 2.2 and Corollary 3.5 hold with $w(u) = (1 \vee u)^{-1}, q(u) = (1 \vee u)^{-2/\alpha}$ and $\mathbb{P}\{X_{r:n}(1) > u\} = n!/(r!(n-r)!)\big(\mathbb{P}\{\zeta_{m,\delta}(1) > u\}\big)^r(1+o(1))$, and (set $m := 1, 0^0 := 1$ if $\delta = 0$)*

$$\mathbb{P}\{\zeta_{m,\delta}(1) > u\} = \delta^{m-1}\frac{2^{1-m/2}}{\Gamma(m/2)}u^{m-2}\exp\left(-\frac{u^2}{2}\right)(1+o(1)),$$

$$\Theta_r(x) = \mathbb{P}\left\{\int_0^{\infty} \mathbb{I}_{(u,\infty)}\big(\min_{1 \leq i \leq r}(\sqrt{2}Z_i(D^{1/\alpha}t) - Dt^{\alpha} + E_i - \beta_4 \kappa t)\big)\, dt > x\right\},$$

where $\beta_4 = 1$ if $\alpha = 1$, and 0 otherwise.

Theorem 4.2. *Let $\{\zeta_{m,\delta}(t), t \geq 0\}$ be defined by (1) to be satisfied by (14) with $\alpha \in (1,2]$. Let $X_{r:n}$ be generated by $\zeta_{m,\delta}, \delta \in [0,1]$. Then Propositions 2.1, 2.2 and Corollary 3.5 hold where $w(u) = (1 \vee u)^{-1}, q(u) = (1 \vee u)^{-2}, \Theta_r(x) = e^{-\kappa r x}$ and $\mathbb{P}\{X_{r:n}(1) > u\}$ is the same as in Theorem 4.1.*

5. Proofs

In this section, we will first give, by verifying Conditions A, B imposed on X holds with the order statistics $X_{r:n}$, respectively, the proofs of Propositions 2.1, 2.2. Then our main results (Theorems 3.1–3.4, Corollary 3.5) follow mainly by the verifications of the tightness Conditions C, C$'$ and C*. We end this section with the proofs of Theorems 4.1, 4.2 concerning the generalized self-similar skew-Gaussian process.

In what follows, we write $\stackrel{d}{=}$ for the equality in distributions and $\stackrel{d}{\to}$ for the convergence in distribution (or the convergence of finite-dimensional distribution if both sides of it are random processes).

Proof of Proposition 2.1: Note that $\{X_{r:n}(t), t \geq 0\}$ is a self-similar process with index κ, and further as $u \to \infty$

$$\frac{\mathbb{P}\{X_{r:n}(s) > u\}}{\mathbb{P}\{X_{r:n}(1) > u\}} = \left(\frac{\mathbb{P}\{X(s) > u\}}{\mathbb{P}\{X(1) > u\}}\right)^r (1 + o(1)) \tag{18}$$

holds uniformly for $s \in (0, 1)$. It follows that the marginal distribution $G_r(x) = \mathbb{P}\{X_{r:n}(1) > x\}$ belongs to the Gumbel MDA with auxiliary function rw. Further since $X_{r:n}$ is a self-similar process with index κ, in view of Proposition 1 in [5], the claim follows. We complete the proof. □

Proof of Proposition 2.2: It follows from Proposition 2.1 that

$$\hat{p}_r(x) = \lim_{u \to \infty} \frac{p(u + x/w_r(u))}{p(u)} = \hat{p}(x/r)$$

exists and continuous for $x > 0$. Further, it follows from Lemma 2 in [5] that $\mathbb{P}\{\xi(t) > x\}$ is continuous at $x = 0$ for each $t \in (0, Q)$. Thus, in view of Theorem 1 in [5], the lower bound follows if we verify that (4) holds for the order statistics process $X_{r:n}$ by taking the limit process $\xi_{r:r}$ at $x_i = 0$.

Indeed, (4) follows for $r = n$ and $m \in \mathbb{N}$, since

$$\begin{aligned}
&\mathbb{P}\left\{X_{n:n}(1 - qt_i) > u, i \leq m \middle| X_{n:n}(1) > u\right\} \\
&\quad = \frac{\mathbb{P}\{X_{n:n}(1 - qt_i) > u, i \leq m, X_{n:n}(1) > u\}}{\mathbb{P}\{X_{n:n}(1) > u\}} \\
&\quad \to \mathbb{P}\left\{\min_{1 \leq j \leq n} \xi_j(t_i) > 0, i \leq m\right\}, \quad u \to \infty. \qquad (19)
\end{aligned}$$

Similarly, (4) holds for all $r < n$ if we show that,

$$\begin{aligned}
\mathbb{P}\left\{X_{r:n}(1 - qt_i) > u, i \leq m \middle| X_{r:n}(1) > u\right\} &= \frac{\mathbb{P}\{X_{r:n}(1 - qt_i) > u, i \leq m, X_{r:n}(1) > u\}}{\mathbb{P}\{X_{r:n}(1) > u\}} \\
&= \frac{\mathbb{P}\{\min_{1 \leq j \leq r} X_j(1 - qt_i) > u, i \leq m, \min_{1 \leq j \leq r} X_j(1) > u\}}{\mathbb{P}\{\min_{1 \leq j \leq r} X_j(1) > u\}}(1 + \Upsilon_r(u)), \qquad (20)
\end{aligned}$$

where $\Upsilon_r(u) \to 0$ holds uniformly for $m \in \mathbb{N}, t_i \in (0, Q)$ as $u \to \infty$.

In the following, we only present the proof for the case that $r = n - 1$ and $m = 1$, the other cases follow by similar arguments. Note that we have by Lemma 3.1 in [19]

$$\mathbb{P}\left\{X_{(n-1):n}(1) > u\right\} = n\mathbb{P}\left\{X_{(n-1):(n-1)}(1) > u\right\}(1 + o(1))$$

and

$$
\begin{aligned}
&\mathbb{P}\left\{X_{(n-1):n}(1-qt_1)>u, X_{(n-1):n}(1)>u\right\}\\
&\quad=\mathbb{P}\left\{X_{(n-1):n}(1-qt_1)>u>X_{n:n}(1-qt_1), X_{n:n}(1)>u\right\}\\
&\quad\quad+\mathbb{P}\left\{X_{n:n}(1-qt_1)>u, X_{(n-1):n}(1)>u>X_{n:n}(1)\right\}\\
&\quad\quad+\mathbb{P}\left\{X_{(n-1):n}(1-qt_1)>u>X_{n:n}(1-qt_1), X_{(n-1):n}(1)>u>X_{n:n}(1)\right\}\\
&\quad\quad+\mathbb{P}\left\{X_{n:n}(1-qt_1)>u, X_{n:n}(1)>u\right\}\\
&\quad=: I_{1u}+I_{2u}+I_{3u}+I_{4u}.
\end{aligned}
$$

Since (4) implies that as $u\to\infty$

$$
\mathbb{P}\left\{X_n(1-qt_1)<u, X_n(1)>u\right\}=\mathbb{P}\left\{X_n(1-qt_1)<u\Big|X_n(1)>u\right\}\mathbb{P}\left\{X_n(1)>u\right\}=o(1),
$$

holds uniformly with $t_1\in(0,Q)$, we have

$$
\begin{aligned}
I_{1u}&=n\mathbb{P}\left\{\min_{1\le j\le n-1}X_j(1-qt_1)>u, \min_{1\le j\le n-1}X_j(1)>u\right\}\mathbb{P}\left\{X_n(1-qt_1)<u, X_n(1)>u\right\}\\
&=n\mathbb{P}\left\{\min_{1\le j\le n-1}X_j(1-qt_1)>u, \min_{1\le j\le n-1}X_j(1)>u\right\}o(1).
\end{aligned}
$$

Similarly, $I_{2u}=n\mathbb{P}\left\{\min_{1\le j\le n-1}X_j(1-qt_1)>u, \min_{1\le j\le n-1}X_j(1)>u\right\}o(1)$. Using further the fact that $\mathbb{P}\left\{X_n(1-qt_1)<u, X_n(1)<u\right\}=1+o(1)$ holds uniformly with $t_1\in(0,Q)$ as $u\to\infty$, we have

$$
\begin{aligned}
I_{3u}&=\sum_{i,i'=1,\ldots,n}\mathbb{P}\left\{\min_{1\le j\le n, j\ne i}X_j(1-qt_1)>u, X_i(1-qt_1)<u, \min_{1\le j'\le n, j'\ne i'}X_{j'}(1)>u, X_{i'}(1)<u\right\}\\
&=n\mathbb{P}\left\{\min_{1\le j\le n-1}X_j(1-qt_1)>u, \min_{1\le j'\le n-1}X_{j'}(1)>u\right\}\mathbb{P}\left\{X_n(1-qt_1)<u, X_n(1)<u\right\}\\
&\quad+\frac{n(n-1)}{2}\mathbb{P}\left\{\min_{1\le j\le n-2}X_j(1-qt_1)>u, \min_{1\le j'\le n-2}X_{j'}(1)>u\right\}\\
&\quad\times\mathbb{P}\left\{X_{n-1}(1-qt_1)<u, X_{n-1}(1)>u\right\}\mathbb{P}\left\{X_n(1-qt_1)>u, X_n(1)<u\right\}\\
&=n\mathbb{P}\left\{\min_{1\le j\le n-1}X_j(1-qt_1)>u, \min_{1\le j'\le n-1}X_{j'}(1)>u\right\}(1+o(1)).
\end{aligned}
$$

Moreover, since (19) implies that as $u\to\infty$

$$
\mathbb{P}\left\{\min_{1\le j\le n-k}X_j(1-qt_1)>u, \min_{1\le j'\le n-k}X_{j'}(1)>u\right\}=\Big(\mathbb{P}\left\{X(1)>u\right\}\Big)^{n-k}O(1),\quad k=0,1,2,
$$

the claim (20) follows for $r=n-1$ and $m=1$. Thus, we complete the proof of the lower bound.

Next, we consider the upper bound. In view of Theorem 1 in [5], it suffices to show that, for sufficiently large u and some $D>0$

$$
\mathbb{P}\left\{X_{r:n}(1-qt)>u|X_{r:n}(0)>u\right\}\le D\mathbb{P}\left\{X(1-qt)>u|X(0)>u\right\}
$$

holds locally uniformly with $t\in(0,Q)$. Indeed, for $r=n$, we have

$$
\mathbb{P}\left\{X_{n:n}(1-qt)>u|X_{n:n}(1)>u\right\}=\Big(\mathbb{P}\left\{X(1-qt)>u|X(1)>u\right\}\Big)^n
$$

and for $r < n$, similar arguments as for (20) yield that

$$
\begin{aligned}
\mathbb{P}\left\{X_{r:n}(1-qt) > u \Big| X_{r:n}(1) > u\right\} &= \Big(\mathbb{P}\left\{X(1-qt) > u \Big| X(1) > u\right\}\Big)^r (1+\Upsilon_r(u))\\
&\le D\Big(\mathbb{P}\left\{X(1-qt) > u \Big| X(1) > u\right\}\Big)^r, \quad \text{for large } u
\end{aligned}
$$

holds uniformly for $t \in (0, Q)$ and some positive constant D. Therefore, Condition B holds for the order statistics process $X_{r:n}$. Consequently, the upper bound follows from Theorem 1 in [5]. We complete the proof. □

Proof of Theorem 3.1: In view of Theorem 2 in [5] (note: there was an error that Condition B should be Conditions A, B), it suffices to verify that Conditions A, B are satisfied by $X_{r:n}$, which are already shown in the proof of Proposition 2.2. We conclude the result. □

Proof of Theorem 3.2: In view of Theorem 3 in [5], it suffices to show the tightness Condition C holds for the order statistics $X_{r:n}$, i.e., for given $\sigma > 0$ and $a \in (0, \tilde{a}]$, that

$$
\boldsymbol{v}_r(a,\sigma) \equiv \limsup_{u\to\infty} \frac{\mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t) > u + \sigma/w, \max_{0\le k\le K} X_{r:n}(t_a^u(k)) \le u\right\}}{\mathbb{E}\left(L_r(u)/q\right) + \mathbb{P}\left\{X_{r:n}(1) > u\right\}} < \infty. \tag{21}
$$

Letting $\tilde{u} = u + \sigma/w(u)$ and $\tilde{q} = q(\tilde{u})$, note that the auxiliary function w is self-neglecting (cf. (5)) and (8) holds. It follows from Proposition 1 and Theorem 3 in [5] that

$$
\begin{aligned}
\limsup_{u\to\infty} \frac{\mathbb{P}\left\{\sup_{t\in[0,1]} X(t) > \tilde{u}\right\}}{\mathbb{E}\left(L(u)/q\right) + \mathbb{P}\left\{X(1) > u\right\}} &\le \limsup_{u\to\infty} \frac{\mathbb{P}\left\{\sup_{t\in[0,1]} X(t) > \tilde{u}\right\}}{\mathbb{E}\left(L(\tilde{u})/\tilde{q}\right) + \mathbb{P}\left\{X(1) > \tilde{u}\right\}}\\
\times \limsup_{u\to\infty} \frac{\mathbb{E}\left(L(\tilde{u})/\tilde{q}\right) + \mathbb{P}\left\{X(1) > \tilde{u}\right\}}{\mathbb{E}\left(L(u)/q\right) + \mathbb{P}\left\{X(1) > u\right\}} &\le \left(\frac{2}{\beta_1} + 1\right) D,
\end{aligned}
$$

where D is some constant (which may change line by line below). Using further (12), $\beta_3 > 0$ and the inequality that $(a+b)^n \le 2^n(a^n + b^n), a, b > 0$, we have, for $r = n$

$$
\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,1]} X_{n:n}(t) > \tilde{u},\ \max_{0\le k\le K} X_{n:n}(t_a^u(k)) \le u\right\}\\
&\le \mathbb{P}\left\{\sup_{t\in[0,1]} X_j(t) > \tilde{u}, j \le n, \bigcup_{i=1}^n \{\max_{0\le k\le K} X_i(t_a^u(k)) \le u\}\right\}\\
&\le \sum_{i=1}^n \mathbb{P}\left\{\sup_{t\in[0,1]} X_i(t) > \tilde{u},\ \max_{0\le k\le K} X_i(t_a^u(k)) \le u\right\} \left(\mathbb{P}\left\{\sup_{t\in[0,1]} X(t) > u + \sigma/w\right\}\right)^{n-1}\\
&\le D\boldsymbol{v}(a,\sigma)\bigg(\mathbb{E}\left(L(u)/q\right) + \mathbb{P}\left\{X(1) > u\right\}\bigg)^n \le D\bigg((\mathbb{E}\left(L(u)/q\right))^n + (\mathbb{P}\left\{X(1) > u\right\})^n\bigg)\boldsymbol{v}(a,\sigma)\\
&\le D\Big(\mathbb{E}\left(L_n(u)/q\right) + \mathbb{P}\left\{X_{n:n}(1) > u\right\}\Big)\boldsymbol{v}(a,\sigma) \quad \text{for sufficiently large } u.
\end{aligned}
$$

Next, we present only the proof for $r=n-1$, the other cases follows by similar arguments. Since $\boldsymbol{v}(a,\sigma)$ is finite,

$$
\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,1]} X_{(n-1):n}(t)>\tilde{u},\ \max_{0\le k\le K} X_{(n-1):n}(t_a^u(k))\le u\right\}\\
&\quad\le n\mathbb{P}\left\{\sup_{t\in[0,1]} X_j(t)>\tilde{u}, j\le n-1, \cup_{i,j=1,\ldots,n}\{\max_{0\le k\le K} X_i(t_a^u(k))\le u,\ \max_{0\le k\le K} X_j(t_a^u(k))\le u\}\right\}\\
&\quad\le n\sum_{i,j=1,\ldots,n-1}\mathbb{P}\left\{\sup_{t\in[0,1]} X_i(t)>\tilde{u},\ \max_{0\le k\le K} X_i(t_a^u(k))\le u\right\}\\
&\qquad\times\mathbb{P}\left\{\sup_{t\in[0,1]} X_j(t)>\tilde{u},\ \max_{0\le k\le K} X_j(t_a^u(k))\le u\right\}\left(\mathbb{P}\left\{\sup_{t\in[0,1]} X(t)>\tilde{u}\right\}\right)^{n-3}\\
&\qquad+2n\sum_{i=1,\ldots,n-1,j=n}\mathbb{P}\left\{\sup_{t\in[0,1]} X_i(t)>u+\sigma/w,\ \max_{0\le k\le K} X_i(t_a^u(k))\le u\right\}\left(\mathbb{P}\left\{\sup_{t\in[0,1]} X(t)>\tilde{u}\right\}\right)^{n-2}\\
&\quad\le Dn\big(c_{n-1,2}\boldsymbol{v}^2(a,\sigma)+2(n-1)\boldsymbol{v}(a,\sigma)\big)\Big(\mathbb{E}\left(L(u)/q\right)+\mathbb{P}\left\{X(1)>u\right\}\Big)^{n-1}\qquad\left(\text{set } c_{n,l}=\frac{n!}{l!(n-l)!}\right)\\
&\quad\le D\Big(\mathbb{E}\left(L_{n-1}(u)/q\right)+\mathbb{P}\left\{X_{(n-1):n}(1)>u\right\}\Big)\boldsymbol{v}(a,\sigma).
\end{aligned}
$$

Thus, we obtain that (21) holds for $r=n-1$. We complete the proof. □

Proof of Theorem 3.3: In view of Theorem 4 in [5], it suffices to show that the tightness Condition C$'$ holds for the order statistics $X_{r:n}$, i.e., for each $\sigma>0$

$$
\lim_{a\downarrow 0}\boldsymbol{v}_r(a,\sigma)=0. \tag{22}
$$

In fact, it follows from the similar arguments as for (21) that if $\boldsymbol{v}(a,\sigma)$ is bounded uniformly for $a\in(0,\tilde{a}]$ and $\sigma>0$, then $\boldsymbol{v}_r(a,\sigma)\le D\boldsymbol{v}(a,\sigma)$ with some constant $D>0$ not depending on a. Consequently, it follows that (22) holds and we then finish the proof. □

Proof of Theorem 3.4: In view of Proposition 2 in [5], the fact that the df of $X_{r:n}(1)$ belongs to $D(\Lambda)$ and $\beta_3>0$ imply that Condition B holds for the order process $X_{r:n}$. Further, the proved (20) and (22) show that Conditions A and C$'$ hold for the order process $X_{r:n}$. Therefore, in view of Theorems 5, 6 in [5], we have

$$
\begin{aligned}
&\limsup_{u\to\infty}\frac{1}{\mathbb{E}\left(L_r(u)/q\right)}\mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t)>u\right\}\le\liminf_{x\downarrow 0}\frac{1-\Theta_r(x)}{x}\\
&\quad\le\limsup_{x\downarrow 0}\frac{1-\Theta_r(x)}{x}\le\liminf_{u\to\infty}\frac{1}{\mathbb{E}\left(L_r(u)/q\right)}\mathbb{P}\left\{\sup_{t\in[0,1]} X_{r:n}(t)>u\right\},
\end{aligned}
$$

where

$$
\Theta_r(x)=\int_0^\infty\mathbb{P}\left\{\int_0^Q\mathbb{I}_{(u,\infty)}(\xi_{r:r}(t))\,dt>\frac{x}{\hat{p}(s/r)}\right\}e^{-s}\,ds,\quad x\ge 0.
$$

Further, Theorem 3.1 and Theorem 3.2 show that the limit is positive and finite. □

Proof of Corollary 3.5: Since in view of Proposition 3 (ii) in [5], Condition C* implies that Condition C$'$ holds for the self-similar process X, it suffices to verify that Condition C* holds for the order statistics process $X_{r:n}$, i.e.,

$$
\mathbb{P}\left\{X_{r:n}(1-qt)>u+\frac{\lambda+v}{w}, X_{r:n}(1)\le u+\frac{v}{w}\right\}\le D^*t^d\lambda^{-b}\mathbb{P}\left\{X_{r:n}(1)>u\right\}
$$

for all u large and all $0 < t^\rho \le \lambda \le \lambda_0, v \ge 0, D^* > 0$.

Note that since the process X is self-similar,

$$\mathbb{P}\left\{X(1-qt) > u + (\lambda+v)/w\right\} \le \mathbb{P}\left\{X(1) > u\right\}$$

holds uniformly for $1-qt \in (0,1)$. We have, with the involved constants given as in Condition C*, for $r = n$

$$\begin{aligned}
&\mathbb{P}\left\{X_{n:n}(1-qt) > u + \frac{\lambda+v}{w}, X_{n:n}(1) \le u + \frac{v}{w}\right\}\\
&\quad= \mathbb{P}\left\{X_i(1-qt) > u + \frac{\lambda+v}{w}, i \le n, \bigcup_{j=1,\ldots,n} \{X_j \le u + \frac{v}{w}\}\right\}\\
&\quad\le n\mathbb{P}\left\{X(1-qt) > u + \frac{\lambda+v}{w}, X(1) \le u + \frac{v}{w}\right\}\left(\mathbb{P}\left\{X(1-qt) > u + \frac{\lambda+v}{w}\right\}\right)^{n-1}\\
&\quad\le nDt^d\lambda^{-b}\left(\mathbb{P}\left\{X(1) > u\right\}\right)^n.
\end{aligned}$$

While for $r < n$, we show only the proof for $r = n-1$ and the other cases follow by the similar arguments.

$$\begin{aligned}
&\mathbb{P}\left\{X_{(n-1):n}(1-qt) > u + \frac{\lambda+v}{w}, X_{(n-1):n}(1) \le u + \frac{v}{w}\right\}\\
&\quad= n\mathbb{P}\left\{X_i(1-qt) > u + \frac{\lambda+v}{w}, i \le n-1, \bigcup_{i,j=1,\ldots,n} \{X_i \le u + \frac{v}{w}, X_j \le u + \frac{v}{w}\}\right\}\\
&\quad\le n\sum_{i,j=1,\ldots,n-1}\left(\mathbb{P}\left\{X(1-qt) > u + \frac{\lambda+v}{w}, X(1) \le u + \frac{v}{w}\right\}\right)^2\left(\mathbb{P}\left\{X(1-qt) > u + \frac{\lambda+v}{w}\right\}\right)^{n-3}\\
&\qquad+2n\sum_{i=1,\ldots,n-1,j=n}\mathbb{P}\left\{X_i(1-qt) > u + \frac{\lambda+v}{w}, X_i(1) \le u + \frac{v}{w}\right\}\left(\mathbb{P}\left\{X(1-qt) > u + \frac{\lambda+v}{w}\right\}\right)^{n-2}\\
&\quad\le n\big(c_{n-1,2} + 2(n-1)\big)Dt^d\lambda^{-b}\left(\mathbb{P}\left\{X(1) > u\right\}\right)^{n-1} \le D^*t^d\lambda^{-b}\mathbb{P}\left\{X_{(n-1):n}(1) > u\right\}
\end{aligned}$$

holds for large u and all $0 < t^\rho \le \lambda \le \lambda_0, v \ge 0, D^*$ a positive constant.

Consequently, by Corollary 1 in [5], the desired result follows. □

Proof of Theorem 4.1: Without loss of generality, we assume that $D = 1$. In view of Lemma 3.4 in [19], we obtain that the marginal distribution $G \in D(\Lambda)$ with auxiliary function $w(u) = (1 \vee u)$, and (7)–(10) holds with $\beta_3 = \beta_4 = 1$ if $\alpha = 1$, 0 otherwise. Further, Lemma 3.5 in [19] shows that the f.d.d. of $\{w(\tilde{\zeta}_{m,\delta}(-qt) - u)\Big|(\tilde{\zeta}_{m,\delta}(-qt) > u)$ converges to those of $\sqrt{2}Z(t) - t^\alpha + E$. Thus, in view of Proposition 9 (ii) in [5], Condition A holds with $\xi(t) \overset{d}{=} \sqrt{2}Z(t) - t^\alpha + E - \beta_4\kappa t$.

Further, it follows from Lemma 3.6 in [19] that there exists some positive constant K_p and $p > m$ such that

$$\mathbb{P}\left\{\tilde{\zeta}_{m,\delta}(qt) > u\Big|\tilde{\zeta}_{m,\delta}(0) > u\right\} \quad \le \quad \begin{cases} K_p t^{-\alpha p/2}, & qt \in (0,\epsilon],\\ K_p u^{m-1-p}, & qt \in (\epsilon, T],\end{cases}$$

which together with Proposition 7 in [5] implies that Condition B holds.

In view of Lemma 3.7 in [19], there exist some positve constants D^*, p, λ_0 and $d > 1$ such that

$$\mathbb{P}\left\{\tilde{\zeta}_{m,\delta}(qt) > u + \frac{\lambda}{w}, \tilde{\zeta}_{m,\delta}(0) \le u\right\} \le D^*t^d\lambda^{-p}\mathbb{P}\left\{\tilde{\zeta}_{m,\delta}(0) > u\right\} \tag{23}$$

for $0 < t^{\alpha/2} < \lambda < \lambda_0$ and large u. Thus, it follows from Proposition 2 in [4] and Proposition 8 in [5] that Condition C$'$ is satisfied for the generalized self-similar skew-Gaussian process $\zeta_{m,\delta}$. Therefore, we complete the proof. $\square$

Proof of Theorem 4.2: Since $\alpha > 1$ and $q(u) = (1 \vee u)^{-2}$, Lemma 3.5 in [19] shows that the f.d.d. of $\{w(\tilde{\zeta}_{m,\delta}(-qt) - u)\Big|(\tilde{\zeta}_{m,\delta}(0) > u)$ converges to those of E. Thus, in view of Proposition 9 (ii) in [5], Condition A holds with $\xi(t) \overset{d}{=} E - \kappa t$.

We obtain using (14) that $\mathbb{E}\left(\tilde{\chi}(0)\tilde{\chi}(t)\right) \geq 1 - 2\left|t\right|$ for small t. It further follows from the arguments for Lemma 3.7 in [19] that (23) holds, and thus in view of Proposition 7 in [5] that Condition C$'$ hold. Moreover, since $\beta_3 = 1 > 0$, Condition B follows from Proposition 2 in [5].

Consequently, the desired result follows by a routine calculation giving that $\Theta_r(x) = e^{-\kappa r x}, x > 0$. $\square$

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